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Distributional Chébli–Trimèche transforms ☆

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Abstract

In this paper we investigate the distributional Chébli–Trimèche transforms. We use the so-called kernel method and we are inspired by the papers of Dube and Pandey [L.S. Dube, J.N. Pandey, On the Hankel transform of distributions, Tôhoku Math. J. 27 (1975) 337–354] and Koh and Zemanian [E.L. Koh, A.H. Zemanian, The complex Hankel and I-transformations of generalized functions, SIAM J. Appl. Math. 16 (1968) 945–957] about distributional Hankel transforms. We note that our procedure, supported in a representation of the elements in the corresponding dual spaces, is simpler than the methods described in the above mentioned papers. Some applications of our distributional theory are presented.

Keywords: Chébli–Trimèche transforms; Kernel method; Distributions

1. Introduction

Chébli–Trimèche transforms can be seen as generalized Fourier transforms associated with certain hypergroups defined on $(0,\infty)$ related to some Cauchy problems (see, for instance, [8,16,17]).

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We now recall the main definitions about Chébli–Trimèche setting that will be very useful in the sequel.

We consider the differential operator of second order Δ defined by

$$\Delta = -\frac{d^2}{dx^2} - \frac{A'(x)}{A(x)}\frac{d}{dx} = -\frac{1}{A(x)}\frac{d}{dx}\left(A(x)\frac{d}{dx}\right),$$

where the function A is continuous function on $[0,\infty)$, differentiable on $(0,\infty)$, and satisfies the following conditions [7]:

(i) $A(0) = 0$ and $A(x) > 0$, $x > 0$;
(ii) A is increasing and unbounded;
(iii) $A'(x)/A(x) = (2\alpha+1)/x + B(x)$, $x \in (0,\delta)$, for some $\delta > 0$, and where $\alpha > -1/2$ and B is an odd C^∞-function on $\mathbf{R}$;
(iv) $A'(x)/A(x)$ is a decreasing C^∞-function on $(0,\infty)$. Then there exists

$$\rho = \frac{1}{2}\lim_{x\to+\infty}\frac{A'(x)}{A(x)} \geqslant 0.$$

Usually a function A satisfying the above properties is called a Chébli–Trimèche function.

We also assume that A verifies the following property:

(v) There exist $\eta, x_0 > 0$ such that, for every $x \geqslant x_0$,

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\eta x}\mathcal{C}(x), & \text{if } \rho > 0, \\ \frac{2\alpha+1}{x} + e^{-\eta x}\mathcal{C}(x), & \text{if } \rho = 0, \end{cases}$$

where $\mathcal{C}$ is a C^∞-function such that $d^k\mathcal{C}/dx^k$ is bounded on $[0,\infty)$, for every $k \in \mathbf{N}$.

We denote, for every $\lambda \in \mathbf{C}$, by ψ_λ the solution of the Cauchy problem

$$\Delta u = (\lambda^2 + \rho^2)u \quad \text{on } (0,\infty), \qquad u(0) = 1, \qquad u'(0) = 0. \tag{1.1}$$

The main properties of ψ_λ, $\lambda \in \mathbf{C}$, can by found in [7,16].

The Chébli–Trimèche transform $\mathcal{F}$ associated with Δ is defined by

$$\mathcal{F}(f)(\lambda) = \int_0^\infty \psi_\lambda(y) f(y) A(y)\, dy, \quad \lambda \in \mathbf{R},$$

where, for instance, f is in the Lebesgue space $L_1((0,\infty), A(x)\,dx)$. For $\mathcal{F}$ a Riemann–Lebesgue lemma holds and the inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ is formally given by

$$\mathcal{F}^{-1}(g)(x) = \int_0^\infty \psi_\lambda(x) g(\lambda)\frac{d\lambda}{|c(\lambda)|^2}, \quad x \in (0,\infty),$$

where $c(\lambda)$ is continuous on $[0,\infty)$ and zero free on $]0,\infty)$. In this setting, $c(\lambda)$ can be seen as a Harish-Chandra function. A Plancherel type result for $\mathcal{F}$ holds [6, Theorem 2.2.13]. Moreover, the Chébli–Trimèche transform was investigated in Schwartz function spaces by Bloom and Xu [7]. More recently the authors have studied some aspects about the distributional $\mathcal{F}$ transform and its convolution operation [2–5].

We note that the Hankel [13] and Jacobi [12] transforms appear as special cases of Chébli–Trimèche transforms [16]. The Jacobi transform, for suitable choices of the involved parameters, is just the spherical transform on rank 1 noncompact Riemannian symmetric spaces [10].

In this paper we define the distributional $\mathcal{F}$ transform by using the kernel method inspired in the ideas of Dube and Pandey [9] and Koh and Zemanian [11]. We obtain a representation of the elements of the corresponding dual spaces that allows us to get an explicit form for the distributional $\mathcal{F}$ transform. This paper can be seen as a continuation of [2]. However, we develop here a new idea, supported by the above mentioned representation for the elements of the dual space, that leads us to obtain easier proofs of the main properties of the distributional Chébli–Trimèche transform, in contrast with the proof for the corresponding properties for the distributional Hankel transforms presented in [9] and [11] and for the distributional Chébli–Trimèche transform established in [2]. Our procedure also works in other settings (see, for instance, distributional integral transforms defined in [18]). Finally, as applications of our distributional theory we solve new Dirichlet problems involving the operator Δ and under distributional boundary conditions.

Throughout this paper by C we always denote a suitable positive constant that can be changed from a line to another one.

2. The spaces $\mathcal{H}_\chi$ of functions and their dual

Assume that χ is a continuous function and zero free on $[0,\infty)$ such that $\chi(x) = o(x^{-3})$, as $x \to \infty$. We define the function space $\mathcal{H}_\chi$ as follows. An even C^∞-function ϕ on $\mathbf{R}$ is in $\mathcal{H}_\chi$ if and only if for every $m \in \mathbf{N}$,

$$\alpha_m(\phi) = \sup_{x\in(0,\infty)} \left|\chi(x)\Delta^m\phi(x)\right| < \infty.$$

Note that if ϕ is an even C^∞-function on $\mathbf{R}$ then, for all $m \in \mathbf{N}$, $\Delta^m\phi$ is well defined. The space $\mathcal{H}_\chi$ is endowed with the topology associated with the family $\{\alpha_m\}_{m\in\mathbf{N}}$ of seminorms.

Dube and Pandey [9], Koh and Zemanian [11] and Zemanian [18], amongst others, considered function spaces similar to $\mathcal{H}_\chi$ in their investigations about distributional integral transforms.

Proposition 2.1. *$\mathcal{H}_\chi$ is a Fréchet space. Moreover, the topology of $\mathcal{H}_\chi$ is stronger than the one induced in it by $C^\infty(0,\infty)$.*

Proof. To see these properties we can proceed as in [18, Lemma 6.3-1]. $\square$

If χ is decreasing on $[0,\infty)$ then, by proceeding like in the proof of [2, Proposition 2.1], we can see that the topology of $\mathcal{H}_\chi$ is also generated by the family of seminorms $\{q_m\}_{m\in\mathbf{N}}$, where, for every $m \in \mathbf{N}$,

$$q_m(\phi) = \sup_{x\in(0,\infty)} \left|\chi(x)\frac{d^m}{dx^m}\phi(x)\right|, \quad \phi \in \mathcal{H}_\chi.$$

In [7] Bloom and Xu considered, for every $0 < p \leqslant 2$, the generalized Schwartz space $\mathcal{S}_p$ that is defined as follows. Let $0 < p \leqslant 2$. A function $\phi \in C^\infty(0,\infty)$ is in $\mathcal{S}_p$ if and only if there exists an even function $\varphi \in C^\infty(\mathbf{R})$ such that $\phi = \varphi$ on $(0,\infty)$, and, for every $k, l \in \mathbf{N}$,

$$\mu_{k,l}^p(\phi) = \sup_{x\in(0,\infty)} (1+x)^l \psi_0(x)^{-2/p} \left| \frac{d^k}{dx^k}\phi(x)\right| < \infty.$$

$\mathcal{S}_p$ is equipped with the topology generated by the family $\{\mu_{k,l}^p\}_{k,l\in\mathbf{N}}$ of seminorms. The topology of $\mathcal{S}_p$ is also associated to the system $\{\gamma_{k,l}^p\}_{k,l\in\mathbf{N}}$ of seminorms, where

$$\gamma_{k,l}^p(\phi) = \sup_{x\in(0,\infty)} (1+x)^l \psi_0(x)^{-2/p} \left|\Delta^k\phi(x)\right|,$$

for every $k, l \in \mathbf{N}$ and $\phi \in \mathcal{S}_p$. Moreover, $\phi \in \mathcal{S}_p$ if and only if there exists an even function $\varphi \in C^\infty(\mathbf{R})$ such that $\phi = \varphi$ on $(0,\infty)$, and $\gamma_{k,l}^p(\phi) < \infty$, $k, l \in \mathbf{N}$. The space $\mathcal{S}_p$ coincides when $\rho = 0$ with the space $\mathcal{S}_{\text{even}}$ constituted by the even functions being in the Schwartz space $\mathcal{S}$. Moreover, in [7] the image under the transformation $\mathcal{F}$ of $\mathcal{S}_p$ was characterized when $\rho > 0$ as a space $\mathcal{Q}_p$ of analytic functions. The dual space of $\mathcal{S}_p$ is represented by $\mathcal{S}_p'$.

According to [7, Lemma 3.4(iii)], for every $0 < p \leqslant 2$, the space $\mathcal{S}_p$ is continuously contained in $\mathcal{H}_\chi$. Then, the space $\mathcal{D}_*$ constituted by all those even and C^∞ functions on $\mathbf{R}$ having compact support [16] is contained in $\mathcal{H}_\chi$. The dual space of $\mathcal{D}_*$ is denoted by $\mathcal{D}_*'$.

The dual space of $\mathcal{H}_\chi$ is represented by $\mathcal{H}_\chi'$. If μ is a complex regular Borel measure on $[0,\infty)$ (in particular, μ is bounded) and $k \in \mathbf{N}$, then the functional T defined on $\mathcal{H}_\chi$ by

$$\langle T, \phi\rangle = \int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu(x), \quad \phi \in \mathcal{H}_\chi,$$

is in $\mathcal{H}_\chi'$. In particular, if f is a measurable function on $(0,\infty)$ such that

$$\int_0^\infty \left|\frac{f(x)}{\chi(x)}\right| A(x)\,dx < \infty,$$

then, the functional $\mathcal{L}_f$ defined on $\mathcal{H}_\chi$ through

$$\langle \mathcal{L}_f, \phi\rangle = \int_0^\infty \phi(x) f(x) A(x)\,dx, \quad \phi \in \mathcal{H}_\chi, \tag{2.1}$$

is in $\mathcal{H}_\chi'$.

In the following we establish a representation of the restriction to $\mathcal{D}_*$ of the elements of $\mathcal{H}_\chi'$ that will be very important in the sequel. To prove this one, we proceed by using, in a standard way (see [1,9,15], amongst others), the Hahn–Banach and the Riesz representation theorems.

Proposition 2.2. *Let T be a functional on $\mathcal{H}_\chi$. If $T \in \mathcal{H}'_\chi$, then there exist $r \in \mathbf{N}$ and complex regular Borel measures $\mu_0, \mu_1, \dots, \mu_r$ on $[0, \infty)$ such that*

$$\langle T, \phi\rangle = \sum_{k=0}^{r} \int_0^\infty \chi(x) \Delta^k \phi(x)\, d\mu_k(x), \quad \phi \in \mathcal{D}_*. \tag{2.2}$$

Proof. Assume that $T \in \mathcal{H}'_\chi$. Then, there exist $C > 0$ and $r \in \mathbf{N}$ for which

$$\bigl|\langle T, \phi\rangle\bigr| \leqslant C \max_{0 \leqslant k \leqslant r} \alpha_k(\phi), \quad \phi \in \mathcal{H}_\chi. \tag{2.3}$$

We define the mappings J and S as follows:

$$J : \mathcal{D}_* \to J(\mathcal{D}_*) \subset \prod_{k=0}^{r} M^k, \quad J(\phi) = \bigl(\chi(x)\Delta^k \phi(x)\bigr)_{k=0}^{r}, \quad \phi \in \mathcal{D}_*,$$

where, for every $k = 0, 1, \dots, r$, $M^k = C_0([0, \infty))$, the space of continuous functions on $[0, \infty)$ vanishing in infinity, and

$$S : J(\mathcal{D}_*) \subset \prod_{k=0}^{r} M^k \to \mathbf{C}, \quad S\bigl(\bigl(\chi(x)\Delta^k \phi(x)\bigr)_{k=0}^{r}\bigr) = \langle T, \phi\rangle, \quad \phi \in \mathcal{D}_*.$$

Note that the mapping J is one-to-one. Then, the mapping S is well defined. Moreover, by (2.3), S is continuous when we consider on $J(\mathcal{D}_*)$ the topology induced in it by the product topology of $\prod_{k=0}^{r} M^k$ and on $C_0([0, \infty))$ the usual topology defined by the supremum norm. Hence, the Hahn–Banach theorem allows us to extend S to whole $\prod_{k=0}^{r} M^k$ as an element of $(\prod_{k=0}^{r} M^k)'$, the dual space $\prod_{k=0}^{r} M^k$. Then, according to the Riesz representation theorem [14, Theorem 6.19], we can find complex Borel regular measures $\mu_0, \mu_1, \dots, \mu_r$ on $[0, \infty]$ such that the representation (2.2) holds. $\square$

A property that will be very useful in the sequel is that the measures μ_k, $k = 0, \dots r$, appearing in the representation (2.2) have finite total variation [14, Theorem 6.4]. Then the right-hand side of (2.2) defines an element of $\mathcal{H}'_\chi$. However, since in general $\mathcal{D}_*$ is not dense in $\mathcal{H}_\chi$, the right-hand side of (2.2) does not agree with T for every $\phi \in \mathcal{H}_\chi$.

Because $\mathcal{S}_p$, $0 < p \leqslant 2$, is continuously contained in $\mathcal{H}_\chi$, if $T \in \mathcal{H}'_\chi$ then the restriction of T to $\mathcal{S}_p$ is in $\mathcal{S}'_p$, $0 < p \leqslant 2$.

3. Distributional Chébli–Trimèche transforms

In this section, where the main results will be established, we study the distributional Chébli–Trimèche transforms on $\mathcal{H}'_\chi$.

Firstly we prove that the kernel of the $\mathcal{F}$ transform is in the closure of $\mathcal{D}_*$ in $\mathcal{H}_\chi$.

Proposition 3.1. *Let $\lambda \in \mathbf{C}$ where $|\mathrm{Im}\,\lambda| \leqslant \rho$. Then ψ_λ is in the closure of $\mathcal{D}_*$ in $\mathcal{H}_\chi$.*

Proof. According to [7, Lemma 3.4] and by (1.1), for every $m \in \mathbf{N}$, we can write

$$\begin{aligned}\alpha_m(\psi_\lambda) &= \sup_{x\in(0,\infty)} \left|\chi(x)\Delta^m\psi_\lambda(x)\right| \\ &\leqslant \left(|\lambda|^2+\rho^2\right)^m \sup_{x\in(0,\infty)} \left|\chi(x)\psi_\lambda(x)\right| \\ &\leqslant C\left(|\lambda|^2+\rho^2\right)^m \sup_{x\in(0,\infty)} \left|\chi(x)\right|(1+x)e^{(|\operatorname{Im}\lambda|-\rho)x} < \infty.\end{aligned}$$

Hence $\psi_\lambda \in \mathcal{H}_\chi$.

We now consider a function $\varphi \in \mathcal{D}_*$ such that $\varphi(x)=1$, $|x|\leqslant 1$, and $\phi(x)=0$, $|x|\geqslant 2$, and we define, for every $n\in\mathbf{N}$, $\varphi_n(x)=\varphi(x/n)$, $x\in\mathbf{R}$. It is clear that $\psi_\lambda\varphi_n\in\mathcal{D}_*$. We are going to see that $\psi_\lambda\phi_n\to\psi_\lambda$, as $n\to\infty$, in $\mathcal{H}_\chi$. According to [7, Lemma 4.18(iii)], there exist $\delta, C>0$ such that, for every $n,m\in\mathbf{N}$,

$$\begin{aligned}&\left|\Delta^m(\psi_\lambda\varphi_n-\psi_\lambda)(x)\right| \\ &\quad\leqslant C\sum_{i=1}^{2m}\left|\frac{d^i}{dx^i}\left(\psi_\lambda(x)\varphi_n(x)-\psi_\lambda(x)\right)\right| \\ &\quad\leqslant C\left(\sum_{i=1}^{2m}\sum_{j=1}^{i}\left|\frac{d^{i-j}}{dx^{i-j}}\left(\varphi_n(x)\right)\frac{d^j}{dx^j}\left(\psi_\lambda(x)\right)\right| + \left|\psi_\lambda(x)\varphi_n(x)-\psi_\lambda(x)\right|\right), \quad x\geqslant\delta.\end{aligned}$$

Now, by [7, Lemma 3.6] (by considering $\lambda\in\mathbf{C}$), we obtain that, for every $n,m\in\mathbf{N}$,

$$\left|\chi(x)\Delta^m\left(\psi_\lambda(x)\varphi_n(x)-\psi_\lambda(x)\right)\right| \leqslant C\left|\chi(x)\right|(1+x)^3e^{(|\operatorname{Im}\lambda|-\rho|)x}, \quad x\geqslant\delta.$$

Then, if $\varepsilon>0$ and $m\in\mathbf{N}$ there exists $x_0>0$ for which

$$\left|\chi(x)\Delta^m\left(\psi_\lambda(x)\varphi_n(x)-\psi_\lambda(x)\right)\right|\leqslant\varepsilon, \quad x\geqslant x_0 \text{ and } n\in\mathbf{N}.$$

Moreover, it is clear that if $n\geqslant x_0$, then

$$\chi(x)\Delta^m\left(\psi_\lambda(x)\varphi_n(x)-\psi_\lambda(x)\right)=0, \quad x\in(0,x_0).$$

Thus we conclude that $\alpha_m(\psi_\lambda\varphi_n-\psi_\lambda)\to 0$, as $n\to\infty$, for every $m\in\mathbf{N}$. □

Let $T\in\mathcal{H}'_\chi$. According to Proposition 3.1 we define the Chébli–Trimèche transform $\mathcal{F}T$ of T through

$$(\mathcal{F}T)(\lambda)=\langle T,\psi_\lambda\rangle, \qquad |\operatorname{Im}\lambda|\leqslant\rho.$$

By Proposition 2.2, there exist $r\in\mathbf{N}$ and complex regular Borel measures $\mu_0,\mu_1,\ldots,\mu_r$ on $[0,\infty)$ such that

$$\langle T,\phi\rangle=\sum_{k=0}^{r}\int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu_k(x), \quad \phi\in\mathcal{D}_*. \tag{3.1}$$

Since the two sides of (3.1) define elements of $\mathcal{H}'_\chi$ we have that, for every ϕ being in the closure of $\mathcal{D}_*$ in $\mathcal{H}_\chi$,

$$\langle T,\phi\rangle = \sum_{k=0}^{r}\int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu_k(x).$$

Hence, by (1.1) and Proposition 3.1, we can write

$$\begin{aligned}(\mathcal{F}T)(\lambda) &= \bigl\langle T(x),\psi_\lambda(x)\bigr\rangle\\ &= \sum_{k=0}^{r}\int_0^\infty \chi(x)\Delta^k\psi_\lambda(x)\,d\mu_k(x)\\ &= \sum_{k=0}^{r}\bigl(\lambda^2+\rho^2\bigr)^k\int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu_k(x), \quad |\mathrm{Im}\,\lambda|\leqslant\rho. \end{aligned}\tag{3.2}$$

The representation (3.2) will play a central role in the proof of properties of the distributional Chébli–Trimèche transform on $\mathcal{H}'_\chi$. Indeed, by using representation (3.2) we obtain proofs of these properties that are easier that the corresponding properties of the distributional Hankel transform established in [9,11], where Riemann-sums techniques were often used, and the ones concerning to distributional Chébli–Trimèche transform proved in [2] (see, for instance, the proofs of boundedness and smoothness for the distributional transforms).

We now establish the main properties of the distributional Chébli–Trimèche transform on $\mathcal{H}'_\chi$.

Proposition 3.2. *Let $T\in\mathcal{H}'_\chi$. There exist $l\in\mathbf{N}$ and $C>0$ such that*

$$\bigl|\mathcal{F}(T)(\lambda)\bigr|\leqslant C\bigl(1+|\lambda|^2\bigr)^l,\quad |\mathrm{Im}\,\lambda|\leqslant\rho.$$

Proof. By invoking (3.2), for certain complex regular Borel measures $\mu_0,\mu_1,\ldots,\mu_r$ on $[0,\infty)$, where $r\in\mathbf{N}$, we can write

$$\mathcal{F}(T)(\lambda)=\sum_{k=0}^{r}\bigl(\lambda^2+\rho^2\bigr)^k\int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu_k(x),\quad |\mathrm{Im}\,\lambda|\leqslant\rho.$$

Hence, according to [7, Lemma 3.4], we get

$$\begin{aligned}\bigl|\mathcal{F}(T)(\lambda)\bigr| &\leqslant \sum_{k=0}^{r}\bigl|\lambda^2+\rho^2\bigr|^k\int_0^\infty\bigl|\chi(x)\bigr|e^{(|\mathrm{Im}\,\lambda|-\rho)x}\,d|\mu_k|(x)\\ &\leqslant C\bigl(1+|\lambda|^2\bigr)^r,\quad |\mathrm{Im}\,\lambda|\leqslant\rho.\end{aligned}$$

Here $|\mu_k|$ denotes the total variation of measure μ_k, $k=0,1,\ldots,r$. $\square$

Proposition 3.3. *Let $T \in \mathcal{H}'_\chi$. If $\rho > 0$ then, $\mathcal{F}(T)$ is a holomorphic function on $|\operatorname{Im}\lambda| < \rho$. Moreover, if $l \in \mathbf{N}$, $\mathcal{F}(T)$ is l-times continuously differentiable on $|\operatorname{Im}\lambda| \leqslant \rho$ provided that $\chi(x) = O(x^{-1-l})$, as $x \to \infty$.*

Proof. According to Proposition 2.2 (see the proof of Proposition 3.2) it is sufficient to prove the property when

$$\langle T, \phi\rangle = \int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu(x), \quad \phi \in \mathcal{D}_*,$$

where $k \in \mathbf{N}$ and μ is a complex regular Borel measure on $[0, \infty)$. Then

$$\mathcal{F}(T)(\lambda) = \left(\lambda^2 + \rho^2\right)^k \int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu(x), \qquad |\operatorname{Im}\lambda| \leqslant \rho.$$

Suppose now $\rho > 0$. By [7, Lemma 3.4(iv)], we can differentiate under the integral sign and we can write

$$\frac{d}{d\lambda}\mathcal{F}(T)(\lambda) = 2\lambda k\left(\lambda^2 + \rho^2\right)^{k-1} \int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu(x) + \left(\lambda^2 + \rho^2\right)^k \int_0^\infty \chi(x)\frac{\partial}{\partial\lambda}\psi_\lambda(x)\,d\mu(x), \quad |\operatorname{Im}\lambda| < \rho.$$

Hence $\mathcal{F}(T)$ is holomorphic in the strip $|\operatorname{Im}\lambda| < \rho$.

If $\rho \geqslant 0$ and $\chi(x) = O(x^{-1-l})$, as $x \to \infty$, where $l \in \mathbf{N}$, to see that $\mathcal{F}(T)$ is l-times continuously differentiable it is sufficient to see that the differentiation l-times under the integral sign is, by [7, Lemma 3.4(iv)], justified. □

As it was mentioned in [7, Theorem 4.27], Bloom and Xu characterized the image of the spaces $\mathcal{S}_p$, $0 < p \leqslant 2$, under the Chébli–Trimèche transform. We will write $\mathcal{Q}_p = \mathcal{F}(\mathcal{S}_p)$, $0 < p \leqslant 2$. We need recall the definition of $\mathcal{Q}_p$, $0 < p \leqslant 2$. Let $0 < p \leqslant 2$. We denote by F_p the strip

$$F_p = \left\{\lambda \in \mathbf{C}\colon\ |\operatorname{Im}\lambda| \leqslant \rho(-1 + 2/p)\right\}.$$

A function Φ defined in F_p is in $\mathcal{Q}_p$, if and only if

(i) Φ is even and holomorphic in the interior of F_p, and such that $d^k\Phi/d\lambda^k$ can be continuously extended to F_p, for every $k \in \mathbf{N}$;
(ii) for every $l, k \in \mathbf{N}$,

$$\tau^p_{k,l}(\Phi) = \sup_{\lambda\in F_p}\left(1 + |\lambda|^l\right)\left|\frac{d^k}{d\lambda^k}\Phi(\lambda)\right| < \infty.$$

In the case $\rho = 0$ or $p = 2$, F_p reduces to the real axis and the space $\mathcal{Q}_p$ consists of all those even and C^∞-functions on $\mathbf{R}$ such that (ii) is satisfied, that is, $F_p = \mathcal{S}_{\text{even}}$. The dual space of $\mathcal{Q}_p$ is denoted by $\mathcal{Q}'_p$. Note that if F is a measurable function on $(0,\infty)$ such that $|F(x)| \leqslant C(1+x^l)$, $x \in (0,\infty)$, for some $C>0$ and $l \in \mathbf{N}$, then F defines an element $\mathcal{L}_F$ of $\mathcal{Q}'_p$ by

$$\langle \mathcal{L}_F, \Phi\rangle = \int_0^\infty F(x)\Phi(x)\frac{dx}{|c(x)|^2}, \quad \Phi \in \mathcal{Q}_p. \tag{3.3}$$

Indeed, that $\mathcal{L}_F$ is linear is clear and that $\mathcal{L}_F$ is continuous follows from the boundedness property of F and by [7, (i), p. 92].

The Chébli–Trimèche transform can be defined on the corresponding dual spaces $\mathcal{S}'_p$ and $\mathcal{Q}'_p$ by transposition, that is, if $T \in \mathcal{S}'_p$ then the Chébli–Trimèche transform $\mathcal{F}'(T)$ of T is the element of $\mathcal{Q}'_p$ defined by

$$\langle \mathcal{F}'(T), \Phi\rangle = \langle T, \mathcal{F}^{-1}(\Phi)\rangle, \quad \Phi \in \mathcal{Q}_p, \tag{3.4}$$

where $\mathcal{F}^{-1}$ denotes the inverse of $\mathcal{F}$ and it is given by

$$\mathcal{F}^{-1}(\Phi)(x) = \int_0^\infty \psi_\lambda(x)\Phi(\lambda)\frac{d\lambda}{|c(\lambda)|^2}, \quad x \in (0,\infty) \text{ and } \Phi \in \mathcal{Q}_p.$$

If $T \in \mathcal{H}'_\chi$, then the restriction of T to $\mathcal{S}_p$ is in $\mathcal{S}'_p$, $0 < p \leqslant 2$. Then we can define the distributional Chébli–Trimèche transform $\mathcal{F}'(T)$ of T on $\mathcal{Q}_p$, $0 < p \leqslant 2$, by (3.4). In the following, we prove that $\mathcal{F}'(T)$ and $\mathcal{F}(T)$ coincides as elements of $\mathcal{Q}'_p$, $0 < p \leqslant 2$, where we understand $\mathcal{F}(T)$ defining a functional in $\mathcal{Q}'_p$ by (3.3).

Proposition 3.4. *Let* $0 < p \leqslant 2$ *and* $T \in \mathcal{H}'_\chi$. *Then* $\mathcal{F}(T)$ *defines an element of* $\mathcal{Q}'_p$ *according to* (3.3). *Moreover, for every* $\Phi \in \mathcal{Q}_p$,

$$\langle \mathcal{L}_{\mathcal{F}(T)}, \Phi\rangle = \langle T, \mathcal{F}^{-1}(\Phi)\rangle.$$

Proof. Proposition 3.2 implies that $\mathcal{F}(T)$ defines an element $\mathcal{L}_{\mathcal{F}(T)}$ of $\mathcal{Q}'_p$ by (3.3). Then, we can write

$$\langle \mathcal{L}_{\mathcal{F}(T)}, \Phi\rangle = \int_0^\infty \mathcal{F}(T)(x)\Phi(x)\frac{dx}{|c(x)|^2}.$$

According to Proposition 2.2 it is sufficient to consider T defined on $\mathcal{D}_*$ by

$$\langle T, \phi\rangle = \int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu(x), \quad \phi \in \mathcal{D}_*,$$

where $k \in \mathbf{N}$ and μ is a complex regular Borel measure on $[0,\infty)$. Then

$$\mathcal{F}(T)(\lambda) = \left(\lambda^2+\rho^2\right)^k \int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu(x), \quad |\mathrm{Im}\,\lambda| \leqslant \rho.$$

Hence, we can write

$$\begin{aligned}\langle \mathcal{L}_{\mathcal{F}(T)}, \Phi\rangle &= \int_0^\infty \Phi(\lambda)\big(\lambda^2+\rho^2\big)^k \int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu(x)\frac{d\lambda}{|c(\lambda)|^2}\\ &= \int_0^\infty \chi(x)\Delta^k \int_0^\infty \Phi(\lambda)\psi_\lambda(x)\frac{d\lambda}{|c(\lambda)|^2}\,d\mu(x)\\ &= \int_0^\infty \chi(x)\Delta^k\mathcal{F}^{-1}(\Phi)(x)\,d\mu(x)\\ &= \big\langle T, \mathcal{F}^{-1}(\Phi)\big\rangle, \quad \Phi\in\mathcal{Q}_p.\end{aligned}$$

The interchange in the order of integration is justified by [7, Lemma 3.4 and (i), p. 92] and the properties of χ and $\Phi\in\mathcal{Q}_p$. $\square$

We now prove an inversion formula for the distributional Chébli–Trimèche transform.

Proposition 3.5. *Let $T\in\mathcal{H}'_\chi$. We define, for every $r>0$,*

$$T_r(x) = \int_0^r \mathcal{F}(T)(\lambda)\psi_\lambda(x)\frac{d\lambda}{|c(\lambda)|^2}, \quad x\in(0,\infty).$$

Then, $\langle \mathcal{L}_{T_r}, \phi\rangle \to \langle T, \phi\rangle$, as $r\to\infty$, for every $\phi\in\mathcal{D}_*$.

Proof. Suppose that

$$\langle T, \phi\rangle = \int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu(x), \quad \phi\in\mathcal{D}_*,$$

where $k\in\mathbf{N}$ and μ is a complex regular Borel measure on $[0,\infty)$. Then

$$\mathcal{F}(T)(\lambda) = \big(\lambda^2+\rho^2\big)^k \int_0^\infty \chi(x)\psi_\lambda(x)\,d\mu(x), \quad |\mathrm{Im}\,\lambda|\leqslant\rho.$$

Note that, according to [7, Lemma 3.4], for every $r>0$, T_r is a bounded function on $\mathbf{R}$ and T_r defines the element L_{T_r} of $\mathcal{D}'_*$ by (2.1).

Hence, by interchanging the order of integration, we get

$$\begin{aligned}\langle L_{T_r}, \phi\rangle &= \int_0^\infty T_r(x)\phi(x)A(x)\,dx\\ &= \int_0^\infty \phi(x)\int_0^r \big(\lambda^2+\rho^2\big)^k \int_0^\infty \chi(y)\psi_\lambda(y)\,d\mu(y)\,\psi_\lambda(x)\frac{d\lambda}{|c(\lambda)|^2}A(x)\,dx\end{aligned}$$

$$= \int_0^\infty \chi(y) \int_0^r (\lambda^2 + \rho^2)^k \int_0^\infty \psi_\lambda(x)\phi(x)A(x)\,dx\,\psi_\lambda(y) \frac{d\lambda}{|c(\lambda)|^2}\,d\mu(y)$$
$$= \int_0^\infty \chi(y) \int_0^r \int_0^\infty \psi_\lambda(x)\Delta^k\phi(x)A(x)\,dx\,\psi_\lambda(y) \frac{d\lambda}{|c(\lambda)|^2}\,d\mu(y)$$
$$= \int_0^\infty \chi(y) \int_0^r \psi_\lambda(y)\mathcal{F}(\Delta^k\phi)(\lambda) \frac{d\lambda}{|c(\lambda)|^2}\,d\mu(y), \quad r > 0 \text{ and } \phi \in \mathcal{D}_*.$$

Moreover, by [7, (i), p. 92], since $\mathcal{F}(\Delta^k\phi) \in \mathcal{Q}_p$, for every $\phi \in \mathcal{D}_*$, we have that

$$\lim_{r\to\infty} \int_0^r \psi_\lambda(y)\mathcal{F}(\Delta^k\phi)(\lambda) \frac{d\lambda}{|c(\lambda)|^2} = \Delta^k\phi(y), \quad \phi \in \mathcal{D}_*,$$

uniformly in $y \in (0,\infty)$. Then we conclude that

$$\lim_{r\to\infty} \langle L_{T_r}, \phi\rangle = \int_0^\infty \chi(x)\Delta^k\phi(x)\,d\mu(x) = \langle T, \phi\rangle, \quad \phi \in \mathcal{D}_*. \qquad \square$$

4. Applications

This last section is devoted to present some applications of our distributional Chébli–Trimèche transform to solving differential equations involving the operator Δ.

Next a useful operational rule for distributional Chébli–Trimèche transform $\mathcal{F}$ is established. As usual, the operator Δ is defined on the dual space $\mathcal{H}'_\chi$ by transposition, that is,

$$\langle \Delta T, \phi\rangle = \langle T, \Delta\phi\rangle, \qquad T \in \mathcal{H}'_\chi \text{ and } \phi \in \mathcal{H}_\chi.$$

The operator Δ is continuous from $\mathcal{H}_\chi$ into itself. Then, Δ is also continuous from $\mathcal{H}'_\chi$ into itself. According to (1.1) we can obtain that, for every $T \in \mathcal{H}'_\chi$,

$$\mathcal{F}(\Delta T)(\lambda) = (\lambda^2 + \rho^2)\mathcal{F}(T)(\lambda). \tag{4.1}$$

We now analyze a Dirichlet problem for the operator Δ having a distributional boundary condition. Our objective is to find a solution $u \in \mathcal{H}'_\chi$ of the partial differential equation

$$(-\Delta_x + \partial_t^2)u(x,t) = 0 \tag{4.2}$$

satisfying, for some $0 < p \leqslant 1$, the following boundary conditions:

(i) $\lim_{t\to 0} u(x,t) = T(x)$, in the sense of convergence in $\mathcal{S}'_p$, for certain $T \in \mathcal{H}'_\chi$;
(ii) $\lim_{t\to\infty} u(x,t) = 0$, uniformly in $x \in (0,\infty)$;
(iii) $\lim_{x\to\infty} u(x,t) = 0$, for every $t \in (0,\infty)$; and
(iv) there exists, for every $t \in (0,\infty)$, $\lim_{x\to 0} u(x,t)$.

We firstly obtain the solution u formally and secondly we verify that the result that we have obtained satisfies the Eq. (4.2) and the boundary conditions (i)–(iv).

By (4.1), if $u \in \mathcal{H}'_\chi$ satisfies (4.2), then

$$-\left(\lambda^2+\rho^2\right)U(\lambda,t)+\partial_t^2 U(\lambda,t)=0,$$

where $U(\lambda,t)=\mathcal{F}(u(x,t):x\to\lambda)$. Hence, for every $\lambda\in(0,\infty)$, we have

$$U(\lambda,t)=A(\lambda)e^{-\sqrt{\lambda^2+\rho^2}t}+B(\lambda)e^{\sqrt{\lambda^2+\rho^2}t},\quad t\in(0,\infty).$$

From (ii) it deduces that $B(\lambda)=0$, $\lambda\in(0,\infty)$. Also, (i) leads to

$$A(\lambda)=\mathcal{F}(T)(\lambda),\quad \lambda\in(0,\infty).$$

Then, we conclude that

$$U(\lambda,t)=\mathcal{F}(T)(\lambda)e^{-\sqrt{\lambda^2+\rho^2}t},\quad \lambda,t\in(0,\infty).$$

Now we define the function u on $(0,\infty)\times(0,\infty)$ by

$$u(x,t)=\int_0^\infty \psi_\lambda(x)U(\lambda,t)\frac{d\lambda}{|c(\lambda)|^2},\quad x,t\in(0,\infty).$$

Note that, since $|\psi_\lambda(x)|\leqslant C(1+x)e^{-\rho x}$, $\lambda,x\in(0,\infty)$ [7, Lemma 3.4] and $|c(\lambda)|^{-2}\sim\lambda^{2\alpha+1}$, as $\lambda\to\infty$ [7, (i), p. 92], Proposition 3.2 implies that the last integral is absolutely convergent, and that

$$\left|u(x,t)\right|\leqslant C(1+x)e^{-\rho(x-t/\sqrt{2})},\quad x,t\in(0,\infty).$$

Hence, by using [7, (3.5), p. 93] we deduce that, for each $t\in(0,\infty)$, $u(.,t)$ defines an element of $\mathcal{S}'_p$, that we will continue denoting by u, through

$$\left\langle u(x,t),\phi(x)\right\rangle=\int_0^\infty u(x,t)\phi(x)A(x)\,dx,\quad \phi\in\mathcal{S}_p.$$

According to Proposition 3.4, we can write

$$\left\langle u(x,t),\phi(x)\right\rangle=\int_0^\infty U(\lambda,t)\mathcal{F}(\phi)(\lambda)\frac{d\lambda}{|c(\lambda)|^2},\quad \phi\in\mathcal{S}_p.$$

Let $\phi\in\mathcal{S}_p$. Note that, $|U(\lambda,t)|\leqslant C|\mathcal{F}(T)(\lambda)|$, $t,\lambda\in(0,\infty)$. Then, by using again Proposition 3.2 and [7, (i), p. 92], we deduce that the function

$$\left|\mathcal{F}(T)(\lambda)\right|\mathcal{F}(\phi)(\lambda)\frac{1}{|c(\lambda)|^2}\in L_1(0,\infty).$$

Then, dominated convergence theorem leads to

$$\lim_{t\to 0}\left\langle u(x,t),\phi(x)\right\rangle=\int_0^\infty \mathcal{F}(T)(\lambda)\mathcal{F}(\phi)(\lambda)\frac{d\lambda}{|c(\lambda)|^2}.$$

Hence Proposition 3.4 implies that

$$\lim_{t\to 0}\big\langle u(x,t),\phi(x)\big\rangle = \langle T,\phi\rangle.$$

Thus (i) is established.

By [7, Lemma 3.4] we obtain, for every $t>1$ and $\delta>0$,

$$\begin{aligned}\big|u(x,t)\big| &\leqslant Ce^{-\rho x}\int_0^\infty \big|\mathcal{F}(T)(\lambda)\big|e^{-\sqrt{\lambda^2+\rho^2}t}\frac{d\lambda}{|c(\lambda)|^2}\\ &\leqslant C\left(\int_0^\delta \big|\mathcal{F}(T)(\lambda)\big|e^{-\lambda t}\frac{d\lambda}{|c(\lambda)|^2}+e^{-\delta(t-1)}\int_\delta^\infty \big|\mathcal{F}(T)(\lambda)\big|e^{-\lambda}\frac{d\lambda}{|c(\lambda)|^2}\right).\end{aligned}$$

Then, according to Proposition 3.2 and [7, (i), p. 92] we deduce that

$$\lim_{t\to\infty} u(x,t)=0,$$

uniformly in $x\in(0,\infty)$, and boundary condition (ii) holds.

To see that u satisfies (iii), we need previously to establish the following result that is a Riemann–Lebesgue type lemma for the inverse of the Chébli–Trimèche transform.

Lemma 4.1. *If $\Phi\in L_1((0,\infty),d\lambda/|c(\lambda)|^2)$, then*

$$\lim_{x\to\infty}\int_0^\infty \psi_\lambda(x)\Phi(\lambda)\frac{d\lambda}{|c(\lambda)|^2}=0.$$

Proof. Assume that $\Phi\in\mathcal{Q}_2$. According to [7, Proposition 4.24], $\mathcal{F}^{-1}(\Phi)\in\mathcal{S}_2$. Hence $\lim_{x\to\infty}\mathcal{F}^{-1}(\Phi)(x)=0$. On the other hand, since $|\psi_\lambda(x)|\leqslant 1$, $x,\lambda\in(0,\infty)$, $\mathcal{F}^{-1}$ defines a continuous linear mapping from $L_1((0,\infty),d\lambda/|c(\lambda)|^2)$ into $L^\infty(0,\infty)$. Then, since $\mathcal{D}_*$ is a dense subspace of $L_1((0,\infty),d\lambda/|c(\lambda)|^2)$ and $\mathcal{D}_*$ is contained in $\mathcal{Q}_2$, we conclude that the desired result holds. □

Now, since $U(.,t)\in L_1((0,\infty),d\lambda/|c(\lambda)|^2)$, for every $t\in(0,\infty)$, Lemma 4.1 allows us to deduce that u satisfies (iii).

Finally, by invoking again dominated convergence theorem it follows that

$$\lim_{x\to 0} u(x,t)=\int_0^\infty U(\lambda,t)\frac{d\lambda}{|c(\lambda)|^2},\quad t\in(0,\infty).$$

Thus we conclude that the function u is a solution for our Dirichlet problem.

To finish we note that by proceeding as in [2] or [9] we can solve distributional differential equations of the type

$$P(\Delta)T=S, \tag{4.3}$$

where P is a polynomial, $S\in\mathcal{H}'_\chi$ is prescribed and $T\in\mathcal{H}'_\chi$ is unknown. To solve (4.3), we use the distributional Chébli–Trimèche transform that we have studied.

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